Almost sure convergence vs. convergence in probability: some niceties

The goal of this problem is to develop a deeper understanding of the subtle relationship between almost sure convergence and convergence in probability. We will prove most of the classical results concerning these two modes of convergence, and also present a slightly less known result: the equivalence of almost sure convergence and convergence in probability for series of independent random variables.

Applications of the results established in this problem are ubiquitous in probability (see other problem sets for concrete examples).

Definitions

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ defined on this probability space. All random variables in this problem are assumed to be real-valued, but most reasonings would apply to more general random variables by using the appropriate norms.

We say that $(X_n)_{n\in\mathbb{N}}$ **converges in probability** towards a random variable X ($X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X$) if and only if

$$\forall \epsilon > 0, \lim_{n\to+\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

We say that $(X_n)_{n\in\mathbb{N}}$ **converges almost surely** towards a random variable X ($X_n \xrightarrow[n\to+\infty]{\text{a.s.}} X$) if and only if

$$\mathbb{P}\left(\left\{\omega \in \Omega \,\middle|\, \lim_{n\to+\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Convergence in probability vs. almost sure convergence: the basics

1. Using Lebesgue's dominated convergence theorem, show that if $(X_n)_{n\in\mathbb{N}}$ converges almost surely towards X, then it converges in probability towards X.

Let us consider a sequence of independent random variables $(Z_n)_{n\in\mathbb{N}}$ where $\mathbb{P}(Z_n = 0) = 1 - \frac{1}{n+1}$ and $\mathbb{P}(Z_n = 1) = \frac{1}{n+1}$.

2. Show that $(Z_n)_{n\in\mathbb{N}}$ converges in probability towards $Z = 0$.

3. Use Borel-Cantelli's second lemma to prove that $\mathbb{P}(\limsup_{n\to+\infty} Z_n = 1) = 1$.

4. Does $(Z_n)_{n\in\mathbb{N}}$ converge almost surely towards $Z = 0$?

Completeness

Let us consider the space $L^0(\Omega, \mathcal{A}, \mathbb{P})$ of random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (we identify random variables that are equal almost surely).

For $X, Y \in L^0(\Omega, \mathcal{A}, \mathbb{P})$, we define $d(X, Y) = \mathbb{E}[\min(|X - Y|, 1)]$.

5. Prove that $d(\cdot, \cdot)$ defines a distance on $L^0(\Omega, \mathcal{A}, \mathbb{P})$.

6. Prove that $X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X$ if and only if $\lim_{n\to+\infty} d(X_n, X) = 0$.

Let us consider a Cauchy sequence $(X_n)_{n\in\mathbb{N}}$ in $L^0(\Omega, \mathcal{A}, \mathbb{P})$ equipped with the distance $d(\cdot,\cdot)$, i.e.

$$\forall \epsilon > 0, \exists n \in \mathbb{N}, \forall p, q \geq n, d(X_p, X_q) \leq \epsilon.$$

7. Prove that there exists a subsequence $(X_{\phi(n)})_n$ such that

$$\forall n \in \mathbb{N}, \quad \mathbb{E}\left[\min\left(|X_{\phi(n+1)} - X_{\phi(n)}|, 1\right)\right] \leq \frac{1}{2^n}.$$

8. Deduce that almost surely $\sum_{n=0}^{+\infty} |X_{\phi(n+1)} - X_{\phi(n)}| < +\infty$.

9. Deduce that $(X_{\phi(n)})_n$ converges almost surely towards $X = X_{\phi(0)} + \sum_{n=0}^{+\infty}\left(X_{\phi(n+1)} - X_{\phi(n)}\right)$.

10. Prove that $(X_n)_{n\in\mathbb{N}}$ converges in probability towards X.

11. Deduce that $L^0(\Omega, \mathcal{A}, \mathbb{P})$ equipped with the distance $d(\cdot,\cdot)$ is complete.

Extraction of a subsequence

Let us consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converging in probability towards a random variable X.

12. By using similar arguments as above, prove that there exists a subsequence $(X_{\phi(n)})_n$ that converges almost surely towards X.

Extraction of sub-subsequences and consequences

13. Show that $(X_n)_{n\in\mathbb{N}}$ converges in probability towards X if and only if from any subsequence of $(X_n)_{n\in\mathbb{N}}$ we can extract a subsequence converging almost surely towards X.
Hint: one way is straightforward, the other can be addressed by contradiction.

14. Deduce that if $(X_n)_{n\in\mathbb{N}}$ converges in probability towards X, then for all continuous function f, $(f(X_n))_n$ converges in probability towards $f(X)$. Prove that it is sufficient to have f continuous at point a if $X = a$ almost surely.

15. Deduce also that if $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ converge in probability towards X and Y respectively, then $(X_nY_n)_{n\in\mathbb{N}}$ converges in probability towards XY, and, for all $a, b \in \mathbb{R}$, $(aX_n + bY_n)_{n\in\mathbb{N}}$ converges in probability towards $aX + bY$.

Some additional characterizations

For a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and a random variable X, we define for all $k, m \in \mathbb{N}$ the event $A_k^m((X_n)_{n\in\mathbb{N}}, X) = \left\{|X_k - X| > \frac{1}{2^m}\right\}$. We also define $A^m((X_n)_{n\in\mathbb{N}}, X) = \limsup_{k\to+\infty} A_k^m((X_n)_{n\in\mathbb{N}}, X)$.

16. Show that $\{\omega \in \Omega | X_k(\omega) \text{ does not converge towards } X(\omega)\} = \bigcup_{m\in\mathbb{N}} A^m((X_n)_{n\in\mathbb{N}}, X)$.

17. Deduce that

$$X_n \xrightarrow[n\to+\infty]{\text{a.s.}} X \iff \mathbb{P}\left(\bigcup_{m\in\mathbb{N}} A^m((X_n)_{n\in\mathbb{N}}, X)\right) = 0 \iff \forall m \in \mathbb{N}, \mathbb{P}(A^m((X_n)_{n\in\mathbb{N}}, X)) = 0.$$

Let us consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and a variable X.

18. Prove the following assertion:

$$\forall \epsilon > 0, \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n - X| > \epsilon) \text{ converges } \implies X_n \xrightarrow[n\to+\infty]{\text{a.s.}} X.$$

19. Find a counterexample to the reciprocal.

20. Show however that the reciprocal is true if the random variables of the sequence $(X_n)_{n\in\mathbb{N}}$ are independent.

Let us consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and a random variable X.

21. Show that $A^m((X_n)_{n\in\mathbb{N}}, X) = \bigcap_{n\in\mathbb{N}} \left\{\sup_{k\geq n} |X_k - X| > \frac{1}{2^m}\right\}$.

22. Deduce that $X_n \xrightarrow[n\to+\infty]{\text{a.s.}} X$ if and only if $\sup_{k\geq n} |X_k - X| \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.

23. Deduce that $(X_n)_{n\in\mathbb{N}}$ converges almost surely if and only if $\sup_{k\geq n} |X_k - X_n| \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.
Hint: to prove that $(X_n)_{n\in\mathbb{N}}$ converges almost surely, show that almost surely $(X_n(\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence, or use $\liminf_{n\to+\infty} X_n$ and $\limsup_{n\to+\infty} X_n$.

Series of independent variables

Let us consider a sequence of independent random variables $(X_n)_{n\in\mathbb{N}}$. Let us define $S_n = \sum_{k=0}^{n} X_k$. We assume that $(S_n)_{n\in\mathbb{N}}$ converges in probability towards a random variable S.

Let us consider $\epsilon > 0$.

For $n, N \in \mathbb{N}$ with $N > n$, let us introduce $\tau_{n,N} = \inf\{k \in \{n+1, \ldots, N\}, |S_k - S_n| > \epsilon\}$.

24. Prove that for all $n, N \in \mathbb{N}$ with $N > n$,

$$\mathbb{P}\left(|S_N - S_n| > \frac{\epsilon}{2}\right) \geq \sum_{k=n+1}^{N} \mathbb{P}\left(|S_N - S_k| \leq \frac{\epsilon}{2}, \tau_{n,N} = k\right).$$

25. Deduce that for all $n, N \in \mathbb{N}$ with $N > n$,

$$\mathbb{P}\left(|S_N - S_n| > \frac{\epsilon}{2}\right) \geq \inf_{n<j\leq N} \mathbb{P}\left(|S_N - S_j| \leq \frac{\epsilon}{2}\right) \mathbb{P}\left(\sup_{n<k\leq N} |S_k - S_n| > \epsilon\right).$$

26. Prove then that

$$\forall \alpha > 0, \exists n_0 \in \mathbb{N}, \forall N > n \geq n_0, \mathbb{P}\left(\sup_{n<k\leq N} |S_k - S_n| > \epsilon\right) \leq \alpha.$$

27. Deduce that

$$\forall \alpha > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \mathbb{P}\left(\sup_{k\geq n} |S_k - S_n| > \epsilon\right) \leq \alpha.$$

28. Deduce that $(S_n)_{n\in\mathbb{N}}$ converges almost surely towards S.

The last result states that for a sequence of independent real-valued random variables $(X_n)_{n\in\mathbb{N}}$, the convergence in probability of $\sum_{n\geq 0} X_n$ implies its almost sure convergence. Therefore, the two modes of convergence are equivalent for series of independent random variables.

It is noteworthy that another equivalent mode of convergence for series of independent random variables is that of convergence in distribution. Classical proofs of this fact involve characteristic functions.