Uniform integrability: main theorems and a result by de la Vallée Poussin

The goal of this problem is to explore different modes of convergence in probability theory, specifically convergence in probability and convergence in L^p. In particular, the problem introduces the notion of *uniform integrability*, which allows for a generalization of Lebesgue's dominated convergence theorem. It also presents the characterization of uniform integrability due to the Belgian mathematician Charles-Jean de la Vallée Poussin.

Applications of the results obtained in this problem are ubiquitous in probability (see other problem sets for concrete examples).

Definitions

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ defined on this probability space. All random variables in this problem are assumed to be real-valued, but most reasonings would apply to more general random variables.

We say that $(X_n)_{n\in\mathbb{N}}$ **converges in probability** towards a random variable X ($X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X$) if and only if

$$\forall\epsilon > 0, \lim_{n\to+\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

We say that $(X_n)_{n\in\mathbb{N}}$ **converges in L^p** ($p \in [1, +\infty)$) towards a random variable X ($X_n \xrightarrow[n\to+\infty]{L^p} X$) if and only if

$$\lim_{n\to+\infty} \mathbb{E}\left[|X_n - X|^p\right] = 0.$$

We also recall the classical notion of almost sure convergence: $(X_n)_{n\in\mathbb{N}}$ **converges almost surely** towards a random variable X ($X_n \xrightarrow[n\to+\infty]{\text{a.s.}} X$) if and only if

$$\mathbb{P}\left(\left\{\omega \in \Omega \,\middle|\, \lim_{n\to+\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Convergence in probability vs. convergence in L^p: the basics

In what follows, p will always be a real number in $[1, +\infty)$.

1. Prove Markov's inequality: $\forall a > 0, \forall Z \in L^p, \mathbb{P}(|Z| \geq a) \leq \frac{\mathbb{E}[|Z|^p]}{a^p}$.
2. Deduce that if a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in L^p towards a random variable X, then $(X_n)_{n\in\mathbb{N}}$ converges also in probability towards X.
3. Let us consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}(X_n = 0) = 1 - \frac{1}{2^{np}}$ and $\mathbb{P}(X_n = 2^n) = \frac{1}{2^{np}}$. Show that $(X_n)_{n\in\mathbb{N}}$ converges in probability towards $X = 0$ but not in L^p.

Convergence in probability vs. convergence in L^p: the concept of uniform integrability

We say that a set $\mathfrak{X}$ of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ is **uniformly integrable** if and only if

$$\lim_{a\to+\infty} \sup_{X\in\mathfrak{X}} \mathbb{E}[|X|1_{|X|>a}] = 0.$$

4. Let X be a random variable in L^1. Show that $\mathfrak{X} = \{X\}$ is uniformly integrable.

5. Show that any finite set of L^1 random variables is uniformly integrable.
6. Show that if $\mathfrak{X}$ is such that $\sup_{X \in \mathfrak{X}} |X| \in L^1$ then $\mathfrak{X}$ is uniformly integrable.

Let $\mathfrak{X}$ be uniformly integrable.

7. Show that $\mathfrak{X}$ is bounded in L^1, i.e., $\sup_{X \in \mathfrak{X}} \mathbb{E}[|X|] < +\infty$.
8. Show that $\mathfrak{X}$ is uniformly absolutely continuous, i.e.,

$$\forall \epsilon > 0, \exists \alpha > 0, \forall A \in \mathcal{F}, \mathbb{P}(A) \leq \alpha \Rightarrow \forall X \in \mathfrak{X}, \mathbb{E}[|X| 1_A] \leq \epsilon.$$

Let us now consider a set $\mathfrak{X}$ of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathfrak{X}$ is bounded in L^1 and uniformly absolutely continuous.

9. Show that $\mathfrak{X}$ is uniformly integrable.

Let us now prove the main result of this section, i.e., for any sequence of random variables $(X_n)_{n \in \mathbb{N}}$ in L^p and any random variable X, we have:

$$X_n \xrightarrow[n \to +\infty]{L^p} X \iff X_n \xrightarrow[n \to +\infty]{\mathbb{P}} X \text{ and } \{|X_n|^p | n \in \mathbb{N}\} \text{ is uniformly integrable.}$$

We start with the case $p = 1$. Let us therefore consider a sequence of random variables $(X_n)_{n \in \mathbb{N}}$ in L^1 and a random variable X.

10. Show that if $X_n \xrightarrow[n \to +\infty]{L^1} X$ then X is in L^1. Deduce that $\{|X_n| | n \in \mathbb{N}\}$ is bounded in L^1.
11. Show that if $X_n \xrightarrow[n \to +\infty]{L^1} X$ then $\{|X_n| | n \in \mathbb{N}\}$ is uniformly absolutely continuous.
12. Conclude for the $\Rightarrow$ part of the above result when $p = 1$.
13. Let us now assume that the sequence $(X_n)_{n \in \mathbb{N}}$ converges in probability towards X and that $\{|X_n| | n \in \mathbb{N}\}$ is uniformly integrable. Prove that $(X_n)_{n \in \mathbb{N}}$ converges in L^1 towards X (this is the $\Leftarrow$ part of the above result when $p = 1$).

Let us now consider the general case $p \in [1, +\infty)$.

14. Prove that $\forall x, y \in \mathbb{R}, |x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$.
15. Use the above inequality and the result for $p = 1$ to obtain the general result.

Complements on uniform integrability: the characterization of de la Vallée Poussin

Let us consider a measurable function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x \to +\infty} \varphi(x) = +\infty$. Let us consider a set $\mathfrak{X}$ of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\sup_{X \in \mathfrak{X}} \mathbb{E}[|X| \varphi(|X|)] < +\infty$.

16. Prove that $\mathfrak{X}$ is bounded in L^1.
17. Prove that $\mathfrak{X}$ is uniformly absolutely continuous.
18. Conclude.
19. Deduce that if a sequence of random variables $(X_n)_{n \in \mathbb{N}}$ converges in probability towards a random variable X and if $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|^p] < +\infty$, for some $p \in (1, +\infty)$, then $(X_n)_{n \in \mathbb{N}}$ converges in L^r towards X for all $r \in [1, p)$.
20. Prove the result of previous question directly using Hölder inequality.

Let us now consider a uniformly integrable set $\mathfrak{X}$.

21. Build an increasing sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} a_n = +\infty$ and for all $n \in \mathbb{N}$, $\sup_{X\in\mathfrak{X}} \mathbb{E}(|X|1_{|X|\geq a_n}) \leq \frac{1}{2^n}$.
22. Prove that the function $\varphi : x \in \mathbb{R}_+ \mapsto 1_{x>0}\frac{1}{x}\sum_{n=0}^{\infty}(x - a_n)_+$ is measurable, finite, and such that $\lim_{x\to+\infty} \varphi(x) = +\infty$.
23. Prove that $\sup_{X\in\mathfrak{X}} \mathbb{E}[|X|\varphi(|X|)] < +\infty$.
24. Conclude.

Remark: Another characterization of uniform integrability is Dunford-Pettis theorem. It uses concepts of weak topology.