Probability and arithmetics: the probability to draw coprime numbers

The goal of this problem is to prove the surprising result that the probability of drawing two coprime integers uniformly at random from the set $\{1, \ldots, N\}$ tends to $\frac{6}{\pi^2}$ as $N \to \infty$. In this problem, the student will discover a probabilistic proof of the Basel problem and encounter—perhaps for the first time—one of the most fascinating special functions in number theory: the Möbius function.

A probabilistic proof of the Basel problem

Let X and Y be two independent random variables with probability density function given by

$$g : t \mapsto \frac{2}{\pi} \frac{1}{1+t^2} 1_{t \geq 0}.$$

Let $Z = \frac{X}{Y}$.

1. Prove that Z has probability density function $h : t \mapsto \frac{4}{\pi^2} \frac{-\log(t)}{1-t^2} 1_{t \geq 0}$.
2. By computing $\mathbb{P}(Z \in (0,1))$ in two different ways, prove that

$$\int_0^1 \frac{-\log(t)}{1-t^2} dt = \frac{\pi^2}{8}.$$

3. Prove that for all $n \in \mathbb{N}$, $\int_0^1 -\log(t) t^{2n} dt = \frac{1}{(2n+1)^2}$.
4. Deduce that

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The Möbius function

Let us define the Möbius function $\mu : \mathbb{N}^* \to \{-1, 0, 1\}$ by:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 1, & \text{if } n \text{ is the product of an even number of distinct prime numbers,} \\ -1, & \text{if } n \text{ is the product of an odd number of distinct prime numbers,} \\ 0, & \text{otherwise.} \end{cases}$$

5. Let $n \in \mathbb{N}^*$. Prove that

$$\sum_{d \text{ divides } n} \mu(d) = 1_{n=1}.$$

Hint: prove first that if the decomposition in primes of $n > 1$ writes $n = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ then we have $\sum_{d \text{ divides } n} \mu(d) = \sum_{k=0}^{r} C_r^k (-1)^k$.

6. Prove that

$$\sum_{m=1}^{+\infty} \frac{\mu(m)}{m^2} \sum_{n=1}^{+\infty} \frac{1}{n^2} = 1.$$

Hint: Use Fubini's theorem.

7. Conclude that

$$\sum_{m=1}^{+\infty} \frac{\mu(m)}{m^2} = \frac{6}{\pi^2}.$$

Inclusion-exclusion principle

Let $K \in \mathbb{N}^*$. Let $A_1, ..., A_K$ be K events.

8. Prove that
$$\mathbb{P}\left(\bigcup_{1\leq i\leq K} A_i\right) = \sum_{k=1}^{K} \sum_{1\leq i_1<\ldots<i_k\leq K} (-1)^{k-1}\mathbb{P}\left(\bigcap_{1\leq j\leq k} A_{i_j}\right).$$
Hint: use the induction principle.

The probability to draw coprime numbers

Let $N \in \mathbb{N}^*$. Let $p_1 < \ldots < p_K$ be the prime numbers of $\{1, \ldots, N\}$.

Let us consider n_1 and n_2 two independent random variables with uniform distribution in $\{1, \ldots, N\}$.

Let us consider the event $B_N = \{n_1 \text{ and } n_2 \text{ are coprime numbers}\}$.

For $i \in \{1, \ldots, K\}$ let us also consider the event $A_i = \{p_i \text{ divides both } n_1 \text{ and } n_2\}$.

9. Prove that for all $k \in \{1, \ldots, K\}$ and for all $i_1 < \ldots < i_k$ in $\{1, \ldots, K\}$,
$$\mathbb{P}\left(\bigcap_{1\leq j\leq k} A_{i_j}\right) = \frac{1}{N^2}\left\lfloor \frac{N}{p_{i_1}\cdots p_{i_k}} \right\rfloor^2.$$

10. Deduce that
$$\mathbb{P}(B_N) = 1 + \frac{1}{N^2}\sum_{k=1}^{K} \sum_{1\leq i_1<\ldots<i_k\leq K} (-1)^k \left\lfloor \frac{N}{p_{i_1}\cdots p_{i_k}} \right\rfloor^2 = \frac{1}{N^2}\sum_{m=1}^{N} \mu(m)\left\lfloor \frac{N}{m} \right\rfloor^2.$$

11. Prove that $\lim_{N\to+\infty} \mathbb{P}(B_N) = \frac{6}{\pi^2}$.