The geometric distribution and how it can help to park your car

The goal of this problem is to introduce the geometric distribution and to derive elementary results about it: moments, memoryless property, etc. Then the problem goes on with a toy model about car parking that involves a random variable with geometric distribution.

Remark: The application is inspired by an exercise proposed by Gilles Pagès and Claude Bouzitat.

Definition and first two moments

Let us consider a sequence $(X_n)_{n\in\mathbb{N}}$ of i.i.d. random variables with Bernoulli distribution $\mathcal{B}(p)$ with parameter $p \in (0,1)$.

We consider $N = \inf\{n \in \mathbb{N} | X_n = 1\}$ (N follows a geometric distribution $\mathcal{G}(p)$).[1]

1. Show that $\forall k \in \mathbb{N}, \mathbb{P}(N = k) = p(1-p)^k$.
2. Show that N has moments of all orders.
3. By differentiating the function $x \in [0,1) \mapsto \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, show that

$$\mathbb{E}[N] = \frac{1-p}{p} \quad \text{and} \quad \mathbb{V}[N] = \frac{1-p}{p^2}.$$

The memoryless property and the link with exponential distribution

4. Compute $\mathbb{P}(N \geq k)$ for all $k \in \mathbb{N}$.
5. Deduce that $\mathbb{P}(N \geq n+k | N \geq k) = \mathbb{P}(N \geq n)$ for all $k, n \in \mathbb{N}$.
6. Explain why this is called "memoryless property".
7. Let us consider an $\mathbb{N}$-valued random variable X satisfying

$$\forall n \in \mathbb{N}, \mathbb{P}(X = n) > 0 \quad \text{and} \quad \forall k, n \in \mathbb{N}, \mathbb{P}(X \geq n+k | X \geq k) = \mathbb{P}(X \geq n).$$

   Show that there exists $p \in (0,1)$ such that $X \sim \mathcal{G}(p)$.

Let us consider a random variable Z with an exponential distribution $\mathcal{E}(\lambda)$ (i.e. probability density function $x \mapsto 1_{x>0}\lambda e^{-\lambda x}$ for some $\lambda > 0$).

8. Show that Z satisfies the memoryless property

$$\forall x, y \in \mathbb{R}_+, \mathbb{P}(Z \geq x+y | Z \geq x) = \mathbb{P}(Z \geq y).$$

9. What is the distribution of $\lfloor Z \rfloor$?

An application to optimal strategies for car parking

We now consider a person (Melania) who wants to go to the restaurant in the city center. We assume that there is only one straight road and that the restaurant location corresponds to the coordinate $x = 0$ on that road, while Melania lives at the coordinate $x = -M$ where $M \in \mathbb{N}^*$. Melania can take her car or go by foot, and she wants to minimize the expected distance she walks. Of course, if she decides to take her car, she has to park it at some point. We assume that there is one parking lot at each integer

[1] There are in fact two different definitions of the geometric distribution. Some authors define $\mathcal{G}(p)$ has the distribution of N, as in this problem, or as the distribution of $N+1$. In practice, one simply needs to be careful about the context.

coordinate $x = m \in \{-M, -M+1, \dots, 0, 1, 2, \dots\}$. However, each parking lot can be free or occupied and we assume that the occupancy of parking lots is modeled by i.i.d Bernoulli random variables $\mathcal{B}(p)$ with parameter $p \in (0,1)$ – value 1 for a free parking lot and 0 for an occupied one.

We first study what happens when Melania takes her car. We assume that she drives until the coordinate $x = n \in \{-M, -M+1, \dots, 0, 1, 2, \dots\}$, then takes the first available parking lot in $\{n, n+1, \dots, \}$, and finally walk towards the restaurant.

10. Explain why the distance walked by Melania can be written as $D_n = |n + N|$ where $N \sim \mathcal{G}(p)$.

Let us define $d : x \in \mathbb{R} \mapsto \mathbb{E}[|x + N|]$.

11. Compute $d(n) = \mathbb{E}[D_n]$ for $n \in \mathbb{N}$.
12. Deduce that it is never optimal for Melania to choose a strategy corresponding to $n \in \mathbb{N}^*$.
13. Show that $d(\cdot)$ is a convex function.
14. Show that $\forall x \in \mathbb{R}, d(x) \geq |x| - \frac{1-p}{p}$.
15. Conclude that the set of minimizers of the restriction of $d(\cdot)$ to $\mathbb{Z}$ is $[\underline{n}, \overline{n}] \cap \mathbb{Z}$, where

$$\underline{n} = \min\{n \in \mathbb{Z}_- | d(n+1) \geq d(n)\} \quad \text{and} \quad \overline{n} = \min\{n \in \mathbb{Z}_- | d(n+1) > d(n)\}.$$

16. Show that

$$\forall n \in \mathbb{Z}_-, d(n+1) - d(n) = 2\mathbb{P}(N \geq -n) - 1 = 2(1-p)^{-n} - 1.$$

17. Conclude that

$$\underline{n} = \left\lceil \frac{\log(2)}{\log(1-p)} \right\rceil \quad \text{and} \quad \overline{n} = \left\lfloor \frac{\log(2)}{\log(1-p)} \right\rfloor + 1.$$

18. What is the optimal strategy of Melania when $p > \frac{1}{2}$ if she takes her car?
19. What about the case where $p = \frac{1}{2}$?
20. Show that

$$\forall n \in \mathbb{Z}_-, d(n) = -n + 1 + \frac{2(1-p)^{1-n} - 1}{p}.$$

Let us consider the case where $p = 1 - 2^{-\frac{1}{k}}$ for some $k \in \mathbb{N}^*$.

21. Show that the optimal strategy of Melania, if she takes her car, is to start looking for a parking lot at coordinate $n = -k$ or $n = -k + 1$.
22. Show that $\min_{n \in \mathbb{Z}} d(n) = d(-k) = k$.
23. Deduce that Melania should take her car if $M > k$ and walk if $M < k$. What happens when $M = k$?

Let us consider the case where p is not of the above form.

24. Show that the optimal strategy of Melania, if she takes her car, is to start looking for a parking lot at coordinate $n = \overline{n}$.
25. Show that $\min_{n \in \mathbb{Z}} d(n) = d(\overline{n}) \in (-\overline{n}, -\overline{n} + 1)$.
26. Deduce that Melania should take her car if $M > -\overline{n}$ and walk if $M \leq -\overline{n}$.