From weak to strong: around the law of large numbers

The expression "law of large numbers," coined by Poisson, refers to a broad class of results stating – under various assumptions – that the running average of a sequence of random variables converges (in some sense) to the expected value. The diversity of these results stems from the assumptions made on the random variables (e.g., independence, existence of moments) and the type of convergence considered: we speak of the *weak law of large numbers* when the convergence is in probability, and of the *strong law of large numbers* when the convergence is almost sure.

Typical undergraduate students are often familiar with proofs of the weak law of large numbers for i.i.d. random variables in L^2, and the strong law of large numbers for i.i.d. random variables in L^4. However, although the general result of almost sure and L^1 convergence for i.i.d. random variables in L^1 is frequently stated in undergraduate courses, it is rarely proved in detail.

The goal of this problem is to fill that gap, by providing a simple proof based on Kolmogorov's zero-one law.

Weak law of large numbers for L^2 i.i.d. variables

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of real-valued L^2 i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with (common) expected value and variance denoted respectively by μ and σ^2. Let us denote by $(S_n)_{n\in\mathbb{N}}$ the sequence defined by

$$S_0 = 0 \quad \text{and} \quad \forall n \in \mathbb{N}^*, S_n = \sum_{k=1}^{n} X_k.$$

1. Let $\epsilon > 0$. Prove that for all $n \in \mathbb{N}^*$,

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}.$$

2. Deduce that $\left(\frac{S_n}{n}\right)_{n\in\mathbb{N}^*}$ converges in probability toward μ.

Strong law of large numbers for L^4 i.i.d. variables

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of real-valued L^4 i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with (common) expected value equal to 0. Let us denote by $(S_n)_{n\in\mathbb{N}}$ the sequence defined by

$$S_0 = 0 \quad \text{and} \quad \forall n \in \mathbb{N}^*, S_n = \sum_{k=1}^{n} X_k.$$

3. Prove that for all $n \in \mathbb{N}^*$,

$$\mathbb{E}[S_n^4] = n\mathbb{E}[X_1^4] + 3n(n-1)\mathbb{E}[X_1^2]^2.$$

4. Let $\epsilon > 0$. Show that there exists a constant $C > 0$ such that

$$\mathbb{P}\left(\frac{|S_n|}{n} > \epsilon\right) \leq \frac{C}{n^2\epsilon^4}.$$

5. Use Borel-Cantelli's lemma to deduce that $\left(\frac{S_n}{n}\right)_{n\in\mathbb{N}^*}$ converges almost surely towards 0.

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of real-valued L^4 i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with (common) expected value denoted by μ. Let us denote by $(S_n)_{n\in\mathbb{N}}$ the sequence defined by

$$S_0 = 0 \quad \text{and} \quad \forall n \in \mathbb{N}^*, S_n = \sum_{k=1}^{n} X_k.$$

6. Prove that $\left(\frac{S_n}{n}\right)_{n\in\mathbb{N}^*}$ converges almost surely towards μ.

Strong law of large numbers for L^1 i.i.d. variables

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of real-valued L^1 i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with (common) expected value denoted by μ. Let us denote by $(S_n)_{n\in\mathbb{N}}$ the sequence defined by

$$S_0 = 0 \quad \text{and} \quad \forall n \in \mathbb{N}^*, S_n = \sum_{k=1}^{n} X_k.$$

Let $a > \mu$. Let us introduce for all $m \in \mathbb{N}$:

$$M_m = \max_{0\leq n\leq m}(S_n - na) \quad \text{and} \quad M'_m = \max_{0\leq n\leq m}(S_{n+1} - X_1 - na).$$

7. Show that $\forall m \in \mathbb{N}$, M_m and M'_m are two L^1 random variables with the same distribution.
8. Show that there exist two random variables M and M' with values in $\mathbb{R}\cup\{+\infty\}$ such that $(M_m)_m$ (resp. $(M'_m)_m$) converges almost surely towards M (resp. M').
9. Show that M and M' have the same distribution.
10. Use Kolmogorov's zero-one law to prove that either $M = +\infty$ almost surely or $M < +\infty$ almost surely.
11. Prove that $\forall m \in \mathbb{N}, M_{m+1} = M'_m - \min(a - X_1, M'_m)$.
12. Deduce that $\forall m \in \mathbb{N}, \mathbb{E}[\min(a - X_1, M'_m)] \leq 0$ and $E[\min(a - X_1, M)] = E[\min(a - X_1, M')] \leq 0$.
13. Reason by contradiction to prove that $M < +\infty$ almost surely.
14. Deduce that $\limsup_{n\to+\infty} \frac{S_n}{n} \leq a$ almost surely.
15. Conclude that $\limsup_{n\to+\infty} \frac{S_n}{n} \leq \mu$ almost surely.
16. Prove then the strong law of large numbers: $\left(\frac{S_n}{n}\right)_{n\in\mathbb{N}^*}$ converges almost surely towards μ.

Scheffé's lemma and the convergence in L^1

Let $(Z_n)_n$ be a sequence of nonnegative random variables. Assume that $(Z_n)_n$ converges almost surely towards a random variable Z such that $\mathbb{E}[Z] < +\infty$. Assume also that $\lim_{n\to+\infty} \mathbb{E}[Z_n] = \mathbb{E}[Z]$.

17. Prove that $\lim_{n\to+\infty} \mathbb{E}[\min(Z_n, Z)] = \mathbb{E}[Z]$.
18. Show that $\forall x, y \in \mathbb{R}, |x - y| = x + y - 2\min(x, y)$.
19. Deduce Scheffé's lemma: $(Z_n)_n$ converges in L^1 towards Z.

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of real-valued L^1 i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with (common) expected value denoted by μ.

20. Using the same notations as above, prove that $\left(\frac{S_n}{n}\right)_{n\in\mathbb{N}^*}$ converges in L^1 towards μ. *Hint: consider $(X_n^+)_{n\in\mathbb{N}^*}$ and $(X_n^-)_{n\in\mathbb{N}^*}$.*

Strong law of large numbers with non-L^1 nonnegative random variables

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of nonnegative i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{E}[X_1] = +\infty$.

21. Using the same notations as above, prove that $\left(\frac{S_n}{n}\right)_{n\in\mathbb{N}^*}$ converges almost surely towards $+\infty$. *Hint: consider* $X'_n = \min(X_n, C)$ *for* $C \in \mathbb{R}$.

Strong law of large numbers: L^1 as a necessary condition

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of real-valued i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Using the same notations as above, assume that $\left(\frac{S_n}{n}\right)_{n\in\mathbb{N}^*}$ converges almost surely towards a real-valued random variable.

22. Prove that $\left(\frac{X_n}{n}\right)_{n\in\mathbb{N}^*}$ converges almost surely towards 0.
23. Use the second lemma of Borel-Cantelli to prove that $\sum_n \mathbb{P}(|X_n| \geq n) < +\infty$.
24. Deduce that $\sum_n \mathbb{P}(|X_1| \geq n) < +\infty$ and that $\mathbb{E}[|X_1|] < +\infty$.

(Bonus) Monte-Carlo methods

The law of large numbers is often used to approximate integrals through what is called a Monte-Carlo method. The idea is to see an integral as an expected value and then to approximate that expected value by using the empirical mean associated with an i.i.d. sample.

25. Find the value of $\int_0^1 \frac{dx}{1+x^2}$.
26. Deduce that if $(X_n)_{n\in\mathbb{N}^*}$ is a sequence of i.i.d. $\mathcal{U}(0,1)$ random variables then, almost surely,

$$\lim_{n\to+\infty} \frac{4}{n} \sum_{k=1}^{n} \frac{1}{1+X_k^2} = \pi.$$

27. Illustrate the above with a Python code.

Remark: Monte-Carlo methods are often use to approximate integrals in high dimension, in finance for instance.