The problem of points: Pascal, Fermat and the Chevalier de Méré

The foundations of modern probability theory were established in the 1930s by the Russian mathematician Kolmogorov (Колмогоров), as part of the broader development of the modern theory of integration initiated by Borel, Lebesgue, Radon, and others. Of course, many important questions in probability were posed and solved well before the formulation of Kolmogorov's axioms. The problem proposed below is one of the seminal problems in the history of probability: the *problem of points*. Though its origin dates back to medieval times, it was revived in the 17^{th} century by the writer and salon theorist Chevalier de Méré, and solved by Pascal and Fermat.

In simple terms, the problem of points is the following. Consider two players, for instance Blaise and Pierre, each betting the same amount of money m. They play a game using a fair coin: Blaise earns one point when the coin shows tails, and Pierre earns one point when it shows heads. The winner of the game—who claims the pot (i.e., $2m$)—is the first player to reach n points. However, for exogenous reasons, they are forced to stop the game before either player has reached n points. The question is: *how should they fairly divide the pot?*

The Problem of Points

We propose three different methods to solve this problem. Let $n-p$ and $n-q$ be the respective current scores of Blaise and Pierre when the game is halted (with $p, q \in \{1, \ldots, n\}$). In other words, Blaise still needs p points to win, and Pierre needs q points. Throughout, we will compute the probability $P_{p,q}$ that Blaise would have eventually won the game.

Preliminaries

1. Explain why the number of tosses necessary to determine the winner after the interruption is at most $p+q-1$.

In what follows, in order to model the problem, we introduce $p+q-1$ independent random variables $(X_k)_{1\leq k\leq p+q-1}$ with Bernoulli $\mathcal{B}(\frac{1}{2})$ distribution, where $\{X_k = 1\}$ corresponds to the event "the kth toss after the interruption would have been a tail".[1]

Method I

For $k \in \{1, \ldots, p+q-1\}$, let us denote by B_k the event "Blaise would have won after exactly k tosses (following the interruption)", and by B the event "Blaise would have won".

2. Show that $B = \bigcup_{k=0}^{q-1} B_{p+k}$.
3. For $k \in \{0, \ldots, q-1\}$, show that

$$B_{p+k} = \{X_1 + \ldots + X_{p+k-1} = p-1, X_{p+k} = 1\}.$$

4. For $k \in \{0, \ldots, q-1\}$, deduce that

$$\mathbb{P}(B_{p+k}) = \frac{C_{p+k-1}^{k}}{2^{p+k}}.$$

5. Deduce that

$$P_{p,q} = \sum_{k=0}^{q-1} \frac{C_{p+k-1}^{k}}{2^{p+k}}.$$

[1] Of course, the players might not have tossed $p+q-1$ coins to determine the winner, but we consider the maximum number of tosses, assuming for instance that the players continue to toss just for fun.

6. Let us suppose that $n = 4$. Compute $P_{p,q}$ for $(p, q) \in \{1, 2, 3, 4\}^2$ using the above formula.

Method II

Let $S = \sum_{k=1}^{p+q-1} X_k$.

7. What is the probability distribution of S?
8. Show that $P_{p,q} = \mathbb{P}(S \geq p)$.
9. Deduce that
$$P_{p,q} = \frac{\sum_{k=0}^{q-1} C_{p+q-1}^{p+k}}{2^{p+q-1}}.$$
10. Let us suppose that $n = 4$. Compute $P_{p,q}$ for $(p, q) \in \{1, 2, 3, 4\}^2$ using the above formula.

Method III: recursion and generating function

Let us extend the definition of $P_{p,q}$ to $(p, q) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ as follows:

- $\forall q \in \mathbb{N}^*, P_{0,q} = 1$,
- $\forall p \in \mathbb{N}^*, P_{p,0} = 0$,
- $\forall p, q \in \mathbb{N}^*, P_{p,q}$ is the probability that Blaise would have won had the game continued after the interruption, if Blaise still had p points to win and Pierre q.

11. Show that
$$\forall p, q \in \mathbb{N}^*, P_{p,q} = \frac{P_{p-1,q} + P_{p,q-1}}{2}.$$
12. Compute $P_{p,q}$ for $(p, q) \in \{1, 2, 3, 4\}^2$ using the above recursive formula. Comment.
13. Let us consider the function
$$G : (x, y) \in [0, 1)^2 \mapsto G(x, y) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} P_{p,q} x^p y^q.$$
Explain why G is a well defined C^∞ function.
14. Show that
$$\forall (x, y) \in [0, 1)^2, G(x, y) = \frac{1}{2} x G(x, y) + \frac{1}{2} y G(x, y) + \frac{1}{2} \frac{xy}{1 - y}.$$
Deduce the expression of G.
15. By using a Taylor expansion of $(x, y) \in [0, 1)^2 \mapsto \frac{1}{2} \frac{xy}{(1-y)\left(1-\frac{1}{2}(x+y)\right)}$, recover an expression for $P_{p,q}$.