Asymptotic expansions of $n!$: from Stirling to Gosper and beyond

The goal of this problem is to derive asymptotic formulas for $n!$. The first steps follow a classical approach, using Wallis integrals to obtain **Stirling formula**. We then introduce two methods to go beyond Stirling's approximation and derive more accurate estimates of $n!$.

The first method, based on a first-order asymptotic expansion using series, leads to **Gosper formula**. The second method uses the **Euler-Maclaurin formula** to derive the full asymptotic expansion of $n!$, extending beyond Stirling's approximation.

Wallis integrals

For $n \in \mathbb{N}$, we define the n-th Wallis integral by

$$W_n = \int_0^{\frac{\pi}{2}} \cos^n(x) dx.$$

1. Show that the sequence $(W_n)_n$ is nonincreasing.
2. By using an integration by parts, show that
$$\forall n \in \mathbb{N}, \quad W_{n+2} = \frac{n+1}{n+2} W_n.$$
3. Deduce that
$$1 - \frac{1}{n+2} \leq \frac{W_{n+1}}{W_n} \leq 1.$$
4. Prove by induction that
$$\forall n \in \mathbb{N}, \quad W_{2n} = \frac{(2n)!}{2^{2n+1}(n!)^2}\pi \quad \text{and} \quad W_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$
5. Deduce that
$$\forall n \in \mathbb{N}, \quad W_n W_{n+1} = \frac{\pi}{2(n+1)}.$$
6. Conclude that
$$\lim_{n\to+\infty} \sqrt{n} W_n = \sqrt{\frac{\pi}{2}}.$$

Stirling formula

For $n \in \mathbb{N}^*$, we define

$$u_n = \frac{n! e^n}{n^{n+\frac{1}{2}}}.$$

7. Show that
$$\forall N \in \mathbb{N}^*, \quad \log(u_N) = 1 + \sum_{n=1}^{N-1} \left(1 - \left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right)\right).$$
8. Deduce that $(u_n)_n$ converges towards a positive number K.
9. Show that
$$\lim_{n\to+\infty} \sqrt{2n} W_{2n} = \frac{\pi}{K}.$$
10. Deduce that $K = \sqrt{2\pi}$.
11. Deduce the Stirling formula:
$$n! \sim_{n\to+\infty} \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Gosper formula

12. By expanding $\log\left(1+\frac{1}{n}\right)$ in powers of $\frac{1}{n}$, show that

$$\forall N \in \mathbb{N}^*, \quad \log(u_N) = 1 + \sum_{n=1}^{N-1}\sum_{k=2}^{\infty}(-1)^{k-1}\frac{k-1}{2k(k+1)}\frac{1}{n^k}.$$

13. Deduce that

$$\forall N \in \mathbb{N}^*, \quad \log(u_N) = \frac{1}{2}\log(2\pi) + \sum_{k=2}^{\infty}(-1)^k\frac{k-1}{2k(k+1)}R_N(k),$$

where $R_N(k) = \sum_{n=N}^{\infty}\frac{1}{n^k}$.

14. By using a comparison between series and integral, show that

$$\forall k \geq 2, \forall N \geq 2, \quad \frac{1}{k-1}\frac{1}{N^{k-1}} \leq R_N(k) \leq \frac{1}{k-1}\frac{1}{N^{k-1}} + \frac{1}{N^k}.$$

15. Deduce that

$$\log(u_N) = \frac{1}{2}\log(2\pi) + \frac{1}{12N} + O\left(\frac{1}{N^2}\right).$$

16. Deduce the Gosper formula:

$$n! = \sqrt{\pi\left(2n+\frac{1}{3}\right)}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n^2}\right)\right).$$

Bernoulli polynomials

17. Show that there exists a unique sequence of polynomials $(B_n)_{n\in\mathbb{N}}$ such that
    - $B_0 = 1$,
    - $\forall n \in \mathbb{N}^*, B_n' = nB_{n-1}$,
    - $\forall n \in \mathbb{N}^*, \int_0^1 B_n(x)dx = 0$.
18. Compute B_0, B_1, B_2, B_3, B_4, B_5, and B_6.
19. Show that $\forall n \geq 2, B_n(1) = B_n(0)$.
20. Show that $\forall n \in \mathbb{N}, B_n(1-X) = (-1)^n B_n(X)$.
21. Deduce that $\forall n \in \mathbb{N}^*, B_{2n+1}(0) = 0$.
22. Compute $B_2(0)$, $B_4(0)$, and $B_6(0)$.

Euler-Maclaurin formula

For this part, let us consider $r \in \mathbb{N}^*$ and f a function in $C^{2r}(\mathbb{R}_+)$.

23. Show by induction that

$$\int_0^1 f(x)dx = \frac{1}{2}(f(0)+f(1)) - \sum_{p=1}^{r}\frac{B_{2p}(0)}{(2p)!}\left(f^{(2p-1)}(1) - f^{(2p-1)}(0)\right) + \frac{1}{(2r)!}\int_0^1 B_{2r}(x)f^{(2r)}(x)dx.$$

24. Deduce that for all $n \in \mathbb{N}^*$,

$$\begin{aligned}\int_{n-1}^{n} f(x)dx &= \frac{1}{2}(f(n-1)+f(n)) - \sum_{p=1}^{r}\frac{B_{2p}(0)}{(2p)!}\left(f^{(2p-1)}(n) - f^{(2p-1)}(n-1)\right)\\ &\quad + \frac{1}{(2r)!}\int_{n-1}^{n} B_{2r}(x-(n-1))f^{(2r)}(x)dx.\end{aligned}$$

25. Deduce the Euler-Maclaurin formula:

$$\forall n \in \mathbb{N}^*, \quad \sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{1}{2}(f(0)+f(n)) + \sum_{p=1}^{r} \frac{B_{2p}(0)}{(2p)!}\left(f^{(2p-1)}(n) - f^{(2p-1)}(0)\right) - \frac{1}{(2r)!}\int_0^n B_{2r}(x - \lfloor x \rfloor) f^{(2r)}(x)dx,$$

where $\lfloor x \rfloor$ is the largest integer smaller than x.

Asymptotic expansion of $n!$

26. By applying Euler-Maclaurin formula to $f : x \in \mathbb{R}_+ \mapsto \log(1+x)$, show that there exists a constant C_r, independent of n, such that

$$\log(n!) = \left(n + \frac{1}{2}\right)\log(n) - n + C_r + \sum_{p=1}^{r} \frac{B_{2p}(0)}{2p(2p-1)} \frac{1}{n^{2p-1}} + O\left(\frac{1}{n^{2r+1}}\right)$$

27. Use Stirling formula to show that $C_r = \frac{1}{2}\log(2\pi)$.
28. Conclude that

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \frac{163879}{209018880n^5} + O\left(\frac{1}{n^6}\right)\right).$$

Key findings:

- Wallis integrals can be computed in closed form by induction.
- Stirling formula:

$$n! \sim_{n\to+\infty} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

- Euler-Maclaurin formula (for f a function in $C^{2r}(\mathbb{R}_+)$ and $(B_k)_k$ the Bernoulli polynomials):

$$\forall n \in \mathbb{N}^*, \quad \sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{1}{2}(f(0)+f(n)) + \sum_{p=1}^{r} \frac{B_{2p}(0)}{(2p)!}\left(f^{(2p-1)}(n) - f^{(2p-1)}(0)\right) - \frac{1}{(2r)!}\int_0^n B_{2r}(x - \lfloor x \rfloor) f^{(2r)}(x)dx.$$